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EFFECTS OF LOCALIZATION AND FORMATION OF STRUCTURES DURING THE COMPRESSION OF A FINITE MASS OF GAS IN A PEAKING MODE*

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The problem of the adiabatic compression of a finite mass of gas with a cylindrical or spherical piston is considered. The pressure at the piston increases in the peaking mode according to the law $P(0,t) = P_0(t_{foc}-t)^{n_g}$, $n_g = -2\gamma(N+1)/(\gamma+1+N(\gamma-1))$, i.e. it becomes infinite as $t \to t_{foc}$, $N = 0, 1, 2$ is the symmetry index and γ is the adiabatic index. The entropy of the gas is distributed over the Lagrangian mass coordinate m: $s = \ln\{a_0|m - m_1|^\delta\}$, a_0, m_1, δ are parameters. The existence of localization of hydrodynamic processes is shown for the case when $N = 0$; in spite of the unlimited growth of pressure at the piston the perturbations do not penetrate beyond a certain finite mass of gas (the region of localization). Outside the region of localization the gas is not affected by the piston and remains in its initial state. The other effect consists of the formation (when $\delta \neq 0$) of gas-dynamic structures, including complex ones such as localized temperature or density maxima connected with the fixed mass of gas.

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In the plane case the piston trajectory ensuring convergence of all characteristics at a single point was found in /1/. Analogous spherical and cylindrical centered compression waves were studied in /2—4/ ** (**See also: Kazhdan Ya.M. On the problem of the adiabatic compression of a gas by means of a spherical piston. Preprint Inst. Prikl. Matem. Akad. Nauk SSSR, Moscow, 89, 1975; Kazhdan Ya.M. Adiabatic compression of a gas by a cylindrical piston, Preprint Inst. Prikl. Matem. Akad. Nauk SSSR, Moscow, 56, 1980.). A somewhat different approach to the study of shock-free compression in /5/ was based on constructing the solutions in terms of separable variables (a corresponding solution for the case of isentropic compression was obtained in /6/). This approach, unlike the solutions in characteristics, can be generalized to the case of media with different physical processes /6, 7/. In both approaches shock-free supercompression demands that the pressure at the piston should vary in a peaking mode, i.e. that it should become infinite on approaching the final instant of time.

Below the selfsimilar solutions in separable variables are used to generalize the results of /7/ to the case of the adiabatic (non-isentropic) compression of a finite mass of gas. Two effects, not dealt with earlier, are also studied, namely the localization of the gas-dynamic perturbations and the appearance in the compressed matter of structures which appear as a result of the non-isentropic character of the medium and localization (inertia) of the gas-dynamic processes. The latter combines them with unsteady thermal structures /8—10/. The entropy distrubition in the gas can be caused, for example, by a shock wave moving with non-uniform velocity.

1. Formulation of the problem. One-dimensional adiabatic gas flow is described by the set of equations

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right)=\frac{\partial (r^N u)}{\partial m},\ \frac{\partial u}{\partial t}=-r^N\frac{\partial p}{\partial m},\quad \frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right)=0,\quad \frac{\partial r}{\partial t}=u \tag{1.1}$$

where m is the Lagrangian mass coordinate, p, ρ, u, r and the pressure, density, velocity and the spatial coordinate, and γ, N are the adiabatic and symmetry indices.

We will choose the adiabaticity integral in the form of a power function

$$p\rho^{-\gamma}=a_0\mid m-m_1\mid^\delta \tag{1.2}$$

(δ, m_1 are parameters). Formula (1.2) holds, in particular, for the entropy distribution behind a shock wave front converging on the centre of symmetry, with $m_1=0,\ \delta=-2(1-\alpha)/(3\alpha)$, where α is the selfsimilarity index (for $\gamma={}^5/_3\ \alpha=0.667$ /11/).

The solution of system (1.1) in separable variables has the form

$$p(m,t)=p_0(t_{foc}-t)^n\,\pi(\xi),\ u(m,t)=u_0(t_{foc}-t)^{n_1}v(\xi) \tag{1.3}$$

where $\pi(\xi)$ and $v(\xi)$ are dimensionless functions of the selfsimilar coordinate

$$\xi=m/m_0 \tag{1.4}$$

The parameters m_0 (with dimension of mass) and u_0 depend on the constant p_0 (determined from the pressure mode at the piston) and the entropy constant a_0* (*See also: Demidov M.A. and Mikhailov A.P. Localization and structures in the adiabatic compression of a finite mass of gas in the peaking mode. Preprint Inst. Prikl. Matem., Akad. Nauk SSSR, Moscow, 8, 1983.).

The following inequalities hold for the indices n, n_1:

$$n=-2\gamma(N+1)/(\gamma+1+N(\gamma-1))<0,\ n_1=-n/(\gamma(N+1))-1<0 \tag{1.5}$$

From (1.3), (1.5) it follows that the gas pressure and velocity are zero at the instant $t=-\infty$, and all its particles are at infinity. When $t\to t_{foc}$ (t_{foc} is the instant of peaking), the pressure, velocity and density of the gas all increase without limit, and the radius tends to zero.

The spatial distribution of the gas-dynamic functions is described by the following system of selfsimilar equations:

$$\frac{d}{d\xi}[(-v)^N v]+|\xi-\xi_1|^{\delta/\gamma}\pi^{-1/\gamma}=0,\ (-v)^N\frac{d\pi}{d\xi}+v=0,\ \xi_1=\frac{m_1}{m_0} \tag{1.6}$$

the boundary conditions for which are given below.

2. Compression of the half-space. We consider the problem of compressing a gas with a flat piston situated at the point $m=0\ (\xi=0)$. The piston velocity is restricted when $t<t_{foc}$, i.e. $0<v(0)<\infty$. Putting $p(0,t)=p_0(t_{foc}-t)^{ns}$ we obtain from (1.3) $\pi(0)=1$.

We seek solutions for which $\pi(\xi), v(\xi)>0$ when $\xi<\xi_f$ ($\xi_f>0$ is the selfsimilar coordinate of the compression wave front separating the moving gas from the unperturbed gas) and when $\xi=\xi_f$, the velocity and pressure of the gas both become zero: $\pi(\xi_f)=v(\xi_f)=0$.

If the quantity ξ_f is finite, we have the localization effect. Indeed, when $m>m_f=\xi_f m_0$, the solution of the problem in question can be continued by means of the steady-state solution of the system (1.1): $\rho(m,t)=\rho_0,\ u(m,t)=p(m,t)=0$. Therefore, when $m<m_f$ (in the region of localization), the gas-dynamic functions increase in the peaking mode, and when $m>m_f$ cold matter at rest can be found (separated from the wave front by a contact discontinuity) into which perturbations do not penetrate.

The solution for the isentropic case ($\delta = m_1 = 0$) can serve as an example:

$$\pi(\xi) = (1 - \xi/\xi_f)^{2\gamma/(\gamma+1)}, \quad v(\xi) = -\pi'(\xi), \ 0 \leqslant \xi \leqslant \xi_f = (\gamma - 1)/\sqrt{2\gamma(\gamma+1)} \tag{2.1}$$

The problem formulated is equivalent to a second-order differential equation (see (1.6)) with $N = 0$), with boundary conditions

$$\pi'' - |\xi - \xi_1|^{\delta/\gamma}\pi^{-1/\gamma} = 0, \ \pi(0) = 1, \ -\infty < \pi'(0) < 0, \quad \pi(\xi_f) = \pi'(\xi_f) = 0 \tag{2.2}$$

The asymptotic forms of the pressure and velocity near the piston and the front $\xi = \xi_f$ are given by the formulas

$$\pi(\xi)_{\xi\to 0} = \begin{cases} 1 - v(0)\xi + |\xi_1|^{\delta/\gamma}\xi^2/2, \ \xi_1 \neq 0, \ v(0) = -\pi'(0) \\ 1 - v(0)\xi + \gamma^2(\delta+\gamma)^{-1}(\delta+2\gamma)^{-1}\xi^{(\delta+2\gamma)/\gamma}, \\ \xi_1 = 0, \ \delta > -\gamma \end{cases} \tag{2.3}$$

When $\xi \to \xi_f$, three cases are possible:

$$\begin{aligned} &\text{a)}\ \pi(\xi) = ((\xi - \xi_f)^2(\xi_f - \xi_1)^{\delta/\gamma}(2\gamma(\gamma-1)^{-1}(\gamma+1)^2)^{\gamma/(\gamma+1)} \times \\ &\quad \xi \to \xi_f \neq \xi_1, \ \infty \\ &\text{b)}\ \pi(\xi) = (\alpha(\alpha-1))^{-\gamma/(\gamma+1)}|\xi - \xi_1|^{\alpha}, \ \alpha = (\delta + 2\gamma)(\gamma + 1), \ \xi \to \xi_f = \xi_1, \ \delta > 1 - \gamma \\ &\text{c)}\ \pi(\xi) = (\alpha(\alpha-1))^{-\gamma/(\gamma+1)}|\xi - \xi_1|^{\alpha}, \ \xi \to \xi_f = \infty, \quad \delta < -2\gamma \end{aligned} \tag{2.4}$$

In cases b) and c) Eqs.(2.4) yield an exact solution for problem (2.2) with

$$|\xi_1| = \xi_1^* = (\alpha(\alpha-1))^{\gamma/(\delta+2\gamma)} \tag{2.5}$$

The entropy singularity cannot be situated within the mass of the compressed gas, i.e. $\xi_1 \geqslant \xi_f$ or $\xi_1 \leqslant 0$. If on the other hand $0 < \xi_1 < \xi_f$, then the density or temperature becomes infinite when $\xi = \xi_1$ (see (1.2)) and a solution cannot be constructed in the region $0 < \xi < \xi_f$.

The substitution

$$y = \pi^{1+1/\gamma}|\xi - \xi_1|^{-2-\delta/\gamma}, \ x = -\pi|\xi - \xi_1|^{-1}/\pi'(\xi) \tag{2.6}$$

makes it possible to study problem (2.2) in the phase space of the following first-order equation:

$$\frac{dy}{dx} = \sigma y^2 x^{-1}\frac{A\alpha x + 1}{y + Axy - x^2}, \quad \sigma = \frac{1}{\gamma} + 1, \ A = \operatorname{sign}(\xi - \xi_1) \tag{2.7}$$

The following point in the xy plane corresponds to the piston:

$$(x(0), y(0)) = (|\xi_1|^{-1}v^{-1}(0), \ |\xi_1|^{-\sigma\alpha}) \tag{2.8}$$

and the values

$$(x_f, y_f) = \begin{cases} (0, 0), \ \xi_f \neq \xi_1, \ \infty \\ (-(A\alpha)^{-1}, |\xi_1^*|^{-\alpha\sigma}), \ \xi_f = \xi_1, \ \xi_f = \infty \end{cases} \tag{2.9}$$

correspond to a point on the front. Problem (2.7)–(2.9) is analysed using standard methods /12/.

Let $\xi_f \neq \xi_1, \infty$ (we consider the solution with a finite front, and the entropy singularity does not coincide with its position). For any γ, δ and ξ_1 there exists a unique integral curve L in the quadrant $x > 0$, $y > 0$, emerging from the point $(0, 0)$ of the front and satisfying the asymptotic forms (2.4), and we have on this curve $y_L = 2\gamma/(\gamma-1)x^2 + \psi(x)$, $\psi(x)x^2 \to 0$ as $x \to 0$. The solution sought is part of the integral curve L lying between the points of the front $(0, 0)$ and the piston (2.8). To construct the solution for given $\xi_1 = y_0^{-1/(\sigma\alpha)}$, we must choose the value of the independent parameter $v(0) = x^{-1}(0)\,y^{1/(\sigma\alpha)}(0)$ so that the point of the piston (2.8) lies on the integral curve L.

Three cases are possible.

1°. $\xi_1 = 0$ (the entropy singularity is situated on the piston $\delta > -\gamma$ /4/, $x(0) = y(0) = +\infty$). The curve L connects the points $(0, 0)$ and $(+\infty, +\infty)$, i.e. a solution exists and is unique since x and y increase monotonically during the motion from the front towards the piston (Fig.1).

2°a. $\xi_1 < 0$, $\alpha \geqslant 0$ (the entropy has no singularities in the gas in front of the piston $x(0)$, $y(0) < +\infty$). The field of integral curves is identical with that of case 1°; therefore a solution exists and is unique.

2°b. $\xi_1 < 0$, $\alpha < 0$ (compared with case 2°a, the entropy falls rapidly on moving away from the piston). When $\alpha < 0$, a singularity $B((-A\alpha)^{-1}, |\xi_1^*|^{-\alpha\sigma})$ appears in the quadrant $x > 0$, $y > 0$, the singularity is a node and becomes a focus when $\alpha < \alpha^*$ (Fig.2) $\alpha^* = (1 - \sqrt{1+\gamma})/2$.

The relations $x < x_1 < \infty$, $y < y_1 < \infty$, $x_1 = x_1(\delta, \gamma)$, $y_1 = y_1(\delta, \gamma)$ hold on curve L. This implies that a solution only exists when $|\xi_1| \leqslant \xi_1^{**} = y_1^{-1/(\sigma\alpha)}$ (which represents a restriction on the entropy distribution) and when $v(0) > v_1^{**} = x_1^{-1}y_1^{1/(\sigma\alpha)}$ (restriction from below on the piston velocity, see (2.6)). When $|\xi_1| < \xi_1^{***} = y_2^{-1/(\sigma\alpha)}$, the solution is unique and ceases to be unique for $\xi_1^{***} < |\xi_1| < \xi_1^{**}$, since for any given value ξ_1 from this interval there exists a spectrum of values $v(0)$, corresponding to various positions of the point of the piston on the integral curve L (Fig.2). When $|\xi_1| \to |\xi_1^*|$, the number of solutions increases without limit and there are infinitely many solutions when $|\xi_1| = \xi_1^*$.

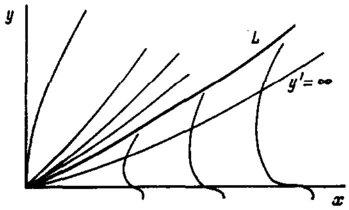

Fig.1

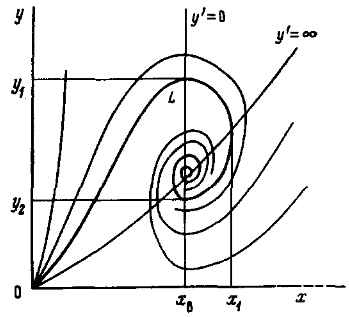

Fig.2

If the point B is a node, then the problem has at most two solutions when $\xi_1^{***} < |\xi_1| < \xi_1^{**}$. In both cases the point B itself is not a solution (see 4° and 5°).

A detailed description of the properties of a solution of type 2°b requires additional explanation.

3°. $\xi_1 > \xi_f$ (the entropy singularity is situated before the compression wave front).

3°a. $\alpha \leqslant 1$. The behaviour of the integral curve L is analogous to the case 2°a; therefore a solution exists and is unique. Unlike the case 2°a, the curve L has an asymptotic form: $y \to +\infty$ as $x \to A^{-1}$, i.e. the piston velocity has a lower limit: $v(0) > A\alpha\,|\xi_1|^{-1}$.

3°b. $\alpha > 1$ (just as in case 2°b, the entropy falls rapidly as the distance from the piston increases). A part of the field of integral curves containing the curve L, and therefore the properties of the solutions, are the same as in the case 2°b.

4°. Let us now consider the case $\xi_f = \xi_1$. Let $\xi_f = \xi_1$ (the entropy singularity is situated at the front, $\delta > 1 - \gamma$ from (2.4)b). A solution exists only when $\xi_1 = \xi_1^*$, is unique and given by formula (2.4) b), and the point B, which represents the centre of focus, corresponds to it in the xy plane.

5°. Finally, if $\xi_f = \infty$ (the infinite mass of gas in front of the piston is compressed, from (2.4) c) we have $\delta < -2\gamma$), then a solution exists only when $\xi_1 = -\xi_1^*$, $\delta < 2\gamma$, is unique and is given by formula (2.4) c). As in the solution 4°, the point B corresponds to it in the xy plane.

In cases 2°b and 3°b the solution of the selfsimilar problem is not unique. For the given values γ, δ, ξ_1 a different number of solutions exists and when $\xi_1 = -\xi_1^*$ ($\xi_1 = \xi_1^*$), we have simultaneously infinitely many solutions of the type 2°b and a solution of the type 5° (or, respectively, a set of solutions of the type 3°b and a solution of the type 4°). However, the solution of the problem of compressing a gas with a piston is unique, since various selfsimilar solutions have various corresponding spatial distributions of the gas-dynamic functions at the instant when the compression commences (which yield equal entropy distribution over the mass of the gas and correspond to the same law of pressure at the piston).

Thus, if the medium (γ) and the entropy distribution in it (δ, ξ_1) are given, we can realize one or another mode of compression (1°—5°). In cases 2°b and 3°b constraints appear on the parameter ξ_1, and in cases 2°b and 3° on the piston velocity $v(0)$. In cases 1°, 4°, 5° the range of variation of the parameter δ is restricted, and in cases 4° and 5° the quantity $|\xi_1| = \xi_1^*$) is also fixed.

The explicit form of system (1.6) implies that the pressure and velocity in the compression wave fall montonically on moving from the piston to the front.

The degree of compression (heating) of the portion of the medium is determined by its entropy and the pressure within it. When the profile in monotonic, we can attain large densities (temperatures) in the compression wave in regions with lower pressure, thanks to the non-isoentropicity, and obtain the gas-dynamic structures, i.e. localized inhomogeneities of the density (temperature) connected with the fixed mass of the gas.

Differentiating the expressions for the density (temperature) $g(\xi) = \pi^{1/\gamma}\,|\xi - \xi_1|^{-\delta/\gamma}$ ($\theta(\xi) = \pi^{1-1/\gamma}\,|\xi - \xi_1|^{\delta/\gamma}$), we obtain the conditions for structures to exist in the compression wave

$$\pi_\xi' = A\delta\pi\,|\xi - \xi_1|^{-1}\ (\pi_\xi' = -A\delta\pi\,|\xi - \xi_1|^{-1}(\gamma - 1)^{-1}) \tag{2.10}$$

In the xy plane conditions (2.10) admit of a descriptive interpretation. The density (temperature) has a structure if the integral curve L intersects the straight line $x = x_1 = -(A\delta)^{-1}$, $(x = x_2 = -x_1/(\gamma - 1))$, and the number of maxima is equal to the number of the points of intersection. Since the straight lines $x = x_1$ and $x = x_2$ lie in different half-planes, we can have either density or temperature structures.

$1°$. $\xi_1 = 0\ (\delta > -\gamma)$. The solution always constains a unique structure (by virtue of the monotonicity of the curve L (a maximum) of the density when $-\gamma < \delta < 0$, or of temperature (when $\delta > 0$). The density $(-\gamma < \delta < 0)$ or temperature $(\delta > 0)$ becomes zero when $\xi = 0\ (m = 0)$.

$2°$a. $\xi_1 < 0, \delta > -2\gamma$. For any $\delta < 0$ (or $\delta > 0$) a parameter ξ_1 can be chosen such, that the integral curve intersects (not more than once) the straight line $x = x_1\ (x = x_2)$; therefore a unique maximum of the density (temperature) may exist.

$2°$b. $\xi_1 < 0, \delta < -2\gamma$. The curve L intersects the straight line $x = x_1 < x_B$ for all values of δ, and the number of points of intersection increases without limit as $\delta \to -\infty$. Therefore the solution can contain an arbitrary number of density structures (maxima or minima).

$3°$a. $\xi_1 > \xi_f, \delta < 1 - \gamma < 0$. The integral curve L intersects the straight line $x = x_2$, therefore ξ_1 can be chosen such that the temperature has a single maximum in the compression wave.

$3°$b. $\xi_1 > \xi_f, \delta > 1 - \gamma$. There are no structures when $1 - \gamma < \delta < 0$, and in the case $\delta > 0$ the solution may contain an arbitrary number of density structures (maxima and minima, just as in $2°$b).

$4°, 5°$. There are no structures; the density and temperatures are monotonic. In case $4°$ the density $g(\xi)$ is constant when $\delta = 2$ (see (2.4), b), i.e. a homogeneous compression of the extremities in the mass of gas in the peaking mode.

Problem (2.2) has a group of analytic solutions of the form $\pi(\xi) = C_1 |\xi - \xi_1|^\nu \times (1 - C_2 |\xi - \xi_1|^\beta)^\theta$ which exist for the following values of δ and γ (the constants C_1 and C_2 are found from the boundary condition on the piston and Eq.(2.2)):

$$\begin{aligned}
&\text{a)}\quad \delta = 0,\ \nu = 0,\ \beta = 1,\ \theta = 2\gamma/(\gamma + 1),\ \xi_1 = 0 \\
&\text{b)}\quad |\xi_1| = \xi_1^*,\ \beta = \theta = 0,\ \nu = \alpha,\ \delta < -2\gamma \text{ or } \delta > 1 - \gamma; \\
&\text{c)}\quad \delta = (1 - 3\gamma)/2,\ \vartheta = 0,\ \beta = 1/\theta,\ \theta = 2\gamma/(\gamma + 1) \\
&\text{d)}\quad \delta = 1 - 3\gamma,\ \beta = -1,\ \nu = 1,\ \theta = 2\gamma/(\gamma + 1); \\
&\text{(e)}\quad \vartheta = \alpha,\ \beta = (2\delta + 3\gamma - 1)/((3\gamma - 1)\gamma/(\gamma + 1) - 1),\quad \theta = \frac{2\gamma}{\gamma + 1},\quad \delta\text{:}\ \left[\frac{\alpha(\alpha - 1)}{2\gamma(\gamma - 1)}\right]^{1/2} = \frac{2\delta + 3\gamma - 1}{(3\gamma - 1)\gamma - (\gamma + 1)}
\end{aligned} \tag{2.11}$$

In case a) the solution is given in Sect.1 (see (1.1), case b) corresponds to solutions $4°$ and $5°$. Cases c)-e) refer to solutions of the type $2°$ and $3°$, and the appearance of structures in them depends on the quantity ξ_1. The analytic solutions (2.11) confirm the general analysis and possess the characteristic properties (localization, structures) of solutions of the problem in question.

In the case of $m_1 = 0\ (\xi_1 = 0)$, the formulation of the problem of the compression of gas by a flat piston in the peaking mode can be generalized to a wider class of selfsimilar solutions (LS- and HS-modes /13–15/. Here in the LS-modes (slower than the S-mode) the gas-dynamic processes are localized and structures appear (when $\delta \neq 0$).

3. Compression in cylindrical and spherical geometry.

When $N = 1, 2$ the gas is inside a cylindrical or spherical piston, collapsing as $t \to t_{foc}$. The mass of gas and its radius are measured from the axis (centre) of symmetry of the system at which the velocity must equal zero. The coordinate $m_p\ (\xi_p = m_p/m_0)$ equal to the mass of compressed gas, corresponds to the piston.

By virtue of the separation of variables (see (1.3)) $v(\xi) = -R(\xi)$, i.e. the velocity becomes zero only at the centre of symmetry of the system $\xi = 0\ (m = 0)$. Thus it follows that we cannot construct solutions with the property of localization within the framework of the selfsimilar formulation given. Therefore we consider, for $N = 1, 2$, also the solutions for which the pressure at $\xi = 0$ is not zero. The boundary conditions for system (1.6) take the form

$$\pi(\xi_p) = 1,\quad |v(\xi_p)| < \infty,\ \pi(0) \geqslant 0,\ v(0) = 0 \tag{3.1}$$

and the last condition of (3.1) is satisfied automatically.

As in the case when $N = 0$, five types of solution may exist depending on the position of the entropy singularity: $1°$. $\xi_1 = \xi_p\ (\xi_1 = 0, N = 0)$; $2°$. $\xi_1 = \xi_p\ (\xi_1 < 0,\ N = 0)$; $3°$. $\xi_1 < 0\ (\xi_1 > \xi_f,\ N = 0)$; $4°$. $\xi_1 = 0\ (\xi_1 = \xi_f, N = 0)$; $5°$. $\xi_p = \infty\ (\xi_f = \infty, N = 0)$.

The solutions of type $5°$ correspond to compression of an infinite mass of gas inside a closed piston, and will not be considered further.

As in the case when $N = 0$, the asymptotic forms of the velocity as $\xi \to \xi_1$ yield, in cases $1°$ and $4°$, the following constraints on the parameter δ: $1°$. $\delta > -\gamma$; $4°$. $\delta > 1 - \gamma$.

When $N = 2$ problem (1.6), (3.1) takes the form

$$\pi'' + (\pi')^4 \pi^{-1/\gamma} |\xi - \xi_1|^{\delta/\gamma} = 0,\ \pi(\xi_p) = 1,\ \pi'(\xi_p) > 0, \tag{3.2}$$
$$\pi'(0) = \infty,\ \pi(0) \geqslant 0$$

The asymptotoic forms of the velocity and pressure as $\xi \to \xi_1 = 0$ imply that solutions of the type 4°, unlike when $N = 0$, exist only when the additional constraint $\delta < 2\gamma$, is imposed, and this represents the special feature of the spherical geometry.

As in the plane case, the substitution

$$y = \pi^{3-1/\gamma} | \xi - \xi_1 |^{\delta/\gamma - 2}/3, \quad x = \pi | \xi - \xi_1 |^{-1}/\pi' (\xi) \tag{3.3}$$

reduces problem (3.2) to that of solving the following first-order equation:

$$\frac{dy}{dx} = \sigma y x \frac{1 + A\alpha x}{x^2 - Ax^3 + y}, \quad \alpha = \frac{\delta - 2\gamma}{3\gamma - 1}, \quad \sigma = 3 - \frac{1}{\gamma} \tag{3.4}$$

If $\pi (0) = 0$, then the point $(0, 0)$ in the xy plane corresponds to the centre of symmetry of the system (see (3.3)). A unique integral curve L emerges from this point, satisfying the asymptotic forms of problem (3.2). If on the other hand the pressure at the centre is not zero, curve L emerges from a fixed point on the axis $y : y_0 = (\pi (0))^{3-1/\gamma} | \xi_1 |^{\delta/\gamma - 2} / 3$. The behaviour of the integral curve L in the xy plane determining the existence and properties of solutions is completely analogous to the case $N = 0$, i.e. solutions of all types $(1^{\circ} - 4^{\circ})$ exist and have the corresponding properties (see Sect.2).

As in the case $N = 0$, problem (3.2) has a set of analytic solutions of the form (2.11)

$$\text{a) } \delta = 0, \quad \nu = 0, \quad \beta = 1, \quad \theta = 2\gamma/(3\gamma - 1), \quad \xi_1 = 0 \tag{3.5}$$

$$\text{b) } | \xi_1 | = \xi_1^* = (3\alpha^2 (\alpha + 1))^{\gamma/(2\gamma - \delta)}, \quad \beta = \theta = 0, \quad \nu = -\alpha$$

$$\text{c) } \delta = (5\gamma - 1)/2, \quad \nu = 0, \quad \beta = (3\gamma - 1)/(2\gamma), \quad \theta = 1/\beta$$

The existence of the density and temperature structures in the solution (3.5) is governed by the parameters δ, γ, ξ_1, ξ_p. The homogeneous compression corresponds to the solution (3.5) b) with $\delta = 2/3$.

When $N = 1$, the second equation in the initial system (1.1) becomes autonomous, and we will write its solution taking (3.1) into account, in the form

$$p (m, t) = p_0 (t_{foc} - t)^{-2} (m_0 + m - m_p)/m_0 \tag{3.6}$$

The pressure profile in the compressed medium does not depend on the medium (γ) itself, nor on the distribution of entropy in it. It is fully defined by the boundary mode at the piston, and this in turn is the same for all media (it is independent of γ).

The general solution of the problem (1.6), (3.1) with $N = 1$ has the form

$$\pi (\xi) = 1 - \xi_p + \xi \tag{3.7}$$

$$v (\xi) = -[\int | \xi - \xi_1 |^{\delta/\gamma} (1 - \xi_p + \xi)^{-1/\gamma} d\xi]^{1/2}$$

$$g (\xi) = | \xi - \xi_1 |^{\delta/\gamma} (1 - \xi_p + \xi)^{-1/\gamma}$$

The integration constant for the velocity is chosen from the condition $v (0) = 0$. Since $\pi (0) = 1 - \xi_p \geqslant 0$, it follows that the mass of the compressed gas has an upper limit $m_p \leqslant m_0$. This inequality represents a generalization of the result in /6/, where a solution of the type (2.1) was also obtained when $\delta = m_1 = 0$ for $N = 0, 1, 2$.

The conditions of existence and the properties of the solutions, and the presence of structures are found directly from (3.7). We note that when $\xi_p = \xi_p^* = 1$, homogeneous compression occurs (see Sect.2) when $\delta = 1$.

The special feature of cylindrical symmetry is the impossibility of constructing a solution of the type 2°b and 3°b, non-unique and containing complex density structures. This is related to the fact that the second equation of the system (1.6) is autonomous.

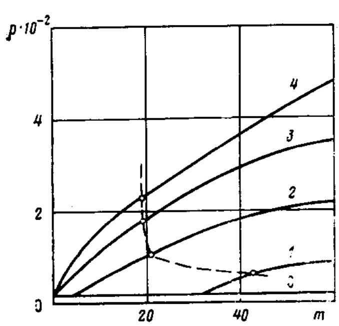

Fig.3

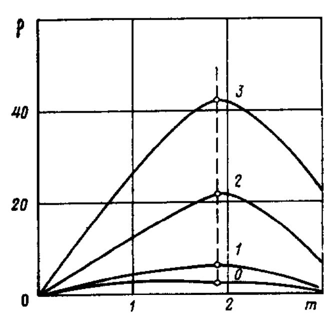

Fig.4

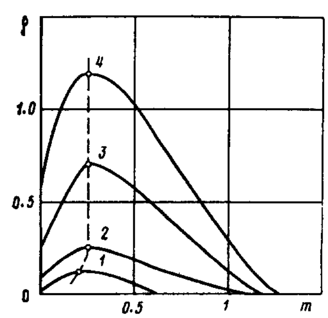

Fig.5

4. Results of numerical calculations.

Numerical calculations carried out for system (1.1) using the *FLORA* /16/ program show, how a selfsimilar compression mode emerges from the non-selfsimilar (e.g. homogeneous) initial data. They also illustrate the stability of the solutions constructed, and of the gas-dynamic structures, and show the method of exciting the structures under monotonic (especially homogeneous) initial data and monotonic boundary mode. The curves 0, 1, 2, 3, 4 shown in Fig.3-5 correspond to the values $t_0 = -1.0$, $a_0 = p_0 = 1$, $t = t_1, t_2, t_3, t_4, t_{foc} = 0$, $\gamma = 5/3$.

Fig.3 illustrates the emergence into the selfsimilar mode from the constant initial data when $N = 2$. The sufficient condition for establishing the selfsimilarity (halting the half-width of the compression wave) is, that the pressure at the piston grows to approximately 100 times its initial value ($m_1 = \delta = 0$, $t_1 = -0.25$, $t_2 = -0.125$, $t_3 = -0.04$, $t_4 = -0.035$).

The stability of the structures is demonstrated by the calculations shown in 4 ($N = 1$). The initial data (including the density maximum) are given in accordance with the analytic solution (3.7) which is reproduced in a stable manner when the pressure at the piston increases by a factor of 10^5 ($\delta = -1.2$, $m_1 = 3.2$, $t_1 = -0.4$, $t_2 = -0.05$, $t_3 = -0.02$).

Fig.5 shows the results of a calculation illustrating the formation of a density structure in an initially homogeneous gas with monotonic boundary conditions, and the localization of the gas-dynamic processes in the case $N = 0$. At the initial instant the pressure at the piston is 50 times greater than the gas pressure. As a result the discontinuity collapses and a shock wave departs from the piston. The entropy behind this shock wave is distributed according to a law resembling (1.2) $m_1 = 0$, $\delta \approx -0.8$. Further, the pressure at the piston varies according the the law (1.3), (1.5) and a compressive S-mode is established. The resulting density structure is stationary with respect to the mass coordinate and increases in the corresponding peaking mode $t_1 = -0.59$, $t_2 = -0.085$, $t_3 = -0.026$, $t_4 = -0.003$.

The peaking modes discussed here have a number of interesting physical properties. In addition to the shock-free, low entropy compression we have the effects of localization and formation of gas-dynamic structures. The localization indicates the fundamental possibility of concentrating and retaining over a finite period of time any amount of energy within some mass of the material, without affecting its remainder. The formation of structures provides adiditional methods of controlling the compression process by creating non-monotonic temperature and density distributions in the compressed material.

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ON THE THEORY OF REGULAR PIECEWISE-HOMOGENEOUS STRUCTURES WITH PIEZOCERAMIC MATRICES*

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A piecewise-homogeneous medium consisting of a piezoceramic matrix bonded by a doubly-periodic system of anisotropic fibres, dielectrics, is considered. The electroelasticity boundary value problems occurring here reduce to a system of Fredholm integral equations of the second kind whose solvability is proved. Concepts of mean mechanical and electrical quantities are introduced from energy considerations, between which a relationship is given by the equations of state of the structure macromodels. The algorithm constructed is realized numerically. Results are presented of computations of the average elastic, electrical, and piezoelectrical properties of the medium as a function of the cell microstructure.

Models of elastic linearly-reinforced composite materials with isotropic and anisotropic components were examined for example, in /1—3/. A survey of the results in the area of electroelasticity boundary value problems can be found in /4/.

1. Formulation of the problem. We consider a transversely isotropic piezoelectric medium (a crystal of the hexagonal 6 mm system, PZT-4, PZT-5, etc. piezoceramic, prepolarized along the z axis), reinforced by a doubly-periodic system of identical anisotropic fibres along the y axis, referred to the crystallographic xyz axes. The fibre transverse cross-section is a simply-connected domain bounded by a simple closed curve l with curvature satisfying the Hölder condition /5/. The fundamental periods of the structure are denoted by ω_1 and $\omega_2\,(\operatorname{Im}(\omega_2/\omega_1)>0)$ the domain occupied by the matrix by D, and the domain occupied by the fibre in the unit cell Π_0 by D_0.

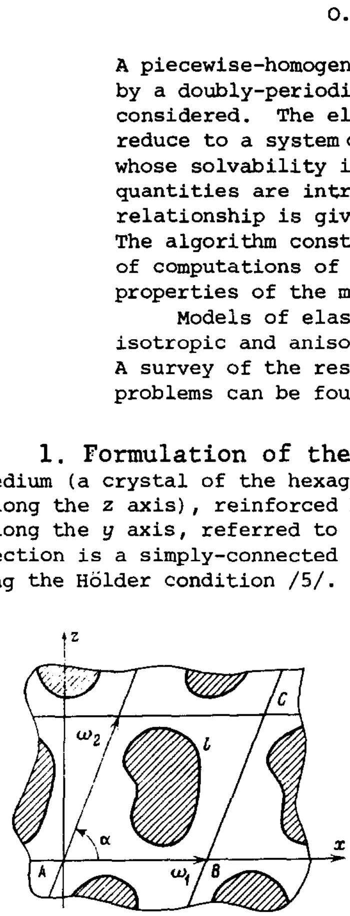

Fig.1

For such an idealization in the plane of the transverse section we obtain an infinitely connected domain that is invariant under the group of translations $T(z)=z+P$, where P is the complex period (Fig.1). We shall assume the mean components of the mechanical stress tensor $\langle\sigma_x\rangle$, $\langle\tau_{xz}\rangle$, $\langle\sigma_z\rangle$ and the electrical intensity vector $\langle E_x\rangle$, $\langle E_z\rangle$ act in the structure.

We will construct a model of a regular piezoceramic medium under the following additional assumptions: a) all fibres have identical physicomechanical properties and possess a plane of elastic symmetry perpendicular to the y axis; b) conditions hold for ideal electrical and mechanical contact between the fibre and the matrix. Under these conditions the fields of the mechanical stresses, the induction and intensity vectors of the electrical field possess the same symmetry group as does the domain D.

The mechanical and electrical quantities in the matrix are defined by the formulas

$$U=2\operatorname{Re}\sum_{k=1}^{3}p_k\Phi_k(z_k),\quad W=2\operatorname{Re}\sum_{k=1}^{3}q_k\Phi_k(z_k) \tag{1.1}$$
$$\sigma_x=2\operatorname{Re}\sum_{k=1}^{3}\gamma_k\mu_k^2\Phi_k'(z_k),\quad \tau_{xz}=-2\operatorname{Re}\sum_{k=1}^{3}\gamma_k\mu_k\Phi_k'(z_k)$$

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